

# T-ADIC EXPONENTIAL SUMS OVER FINITE FIELDS

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ABSTRACT.  $T$ -adic exponential sums associated to a Laurent polynomial  $f$  are introduced. They interpolate all classical  $p^m$ -power order exponential sums associated to  $f$ . The Hodge bound for the Newton polygon of  $L$ -functions of  $T$ -adic exponential sums is established. This bound enables us to determine, for all  $m$ , the Newton polygons of  $L$ -functions of  $p^m$ -power order exponential sums associated to an  $f$  which is ordinary for  $m = 1$ . Deeper properties of  $L$ -functions of  $T$ -adic exponential sums are also studied. Along the way, new open problems about the  $T$ -adic exponential sum itself are discussed.

## 1. INTRODUCTION

**1.1. Classical exponential sums.** We first recall the definition of classical exponential sums over finite fields of characteristic  $p$  with values in a  $p$ -adic field.

Let  $p$  be a fixed prime number,  $\mathbb{Z}_p$  the ring of  $p$ -adic integers,  $\mathbb{Q}_p$  the field of  $p$ -adic numbers, and  $\overline{\mathbb{Q}_p}$  a fixed algebraic closure of  $\mathbb{Q}_p$ . Let  $q = p^a$  be a power of  $p$ ,  $\mathbb{F}_q$  the finite field of  $q$  elements,  $\mathbb{Q}_q$  the unramified extension of  $\mathbb{Q}_p$  with residue field  $\mathbb{F}_q$ , and  $\mathbb{Z}_q$  the ring of integers of  $\mathbb{Q}_q$ .

Fix a positive integer  $n$ . Let  $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$  be a Laurent polynomial in  $n$  variables of the form

$$f(x) = \sum_u a_u x^u, \quad a_u \in \mu_{q-1}, \quad x^u = x_1^{u_1} \cdots x_n^{u_n},$$

where  $\mu_k$  denotes the group of  $k$ -th roots of unity in  $\overline{\mathbb{Q}_p}$ .

**Definition 1.1.** Let  $\psi$  be a locally constant character of  $\mathbb{Z}_p$  of order  $p^m$  with values in  $\overline{\mathbb{Q}_p}$ , and let  $\pi_\psi = \psi(1) - 1$ . The sum

$$S_{f,\psi}(k) = \sum_{x \in \mu_{q^k-1}^n} \psi(\text{Tr}_{\mathbb{Q}_q^k/\mathbb{Q}_p}(f(x)))$$

is called a  $p^m$ -power order exponential sum on the  $n$ -torus  $\mathbb{G}_m^n$  over  $\mathbb{F}_{q^k}$ . The generating function

$$L_{f,\psi}(s) = L_{f,\psi}(s; \mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_{f,\psi}(k) \frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\pi_\psi][[s]]$$

is called the  $L$ -function of  $p^m$ -power order exponential sums over  $\mathbb{F}_q$  associated to  $f(x)$ .

Note that the above exponential sum for  $m \geq 1$  is still an exponential sum over a finite field as we just sum over the subset of roots of unity (corresponding to the elements of a finite field via the Teichmüller lifting), not over the whole finite residue ring  $\mathbb{Z}_q/p^m\mathbb{Z}_q$ . The exponential sum over the whole finite ring  $\mathbb{Z}_q/p^m\mathbb{Z}_q$  and its generating function as  $m$  varies is the subject of Igusa's zeta function, see Igusa [17].

In general, the above  $L$ -function  $L_{f,\psi}(s)$  of exponential sums is rational in  $s$ . But, if  $f$  is non-degenerate, then  $L_{f,\psi}(s)^{(-1)^{n-1}}$  is a polynomial, as was shown in [1, 2] for  $\psi$  of order  $p$ , and in [20] for all  $\psi$ . By a result of [12], if  $p$  is large enough, then  $f$  is generically non-degenerate. For non-degenerate  $f$ , the location of the zeros of  $L_{f,\psi}(s)^{(-1)^{n-1}}$  becomes an important issue. The  $p$ -adic theory of such  $L$ -functions was developed by Dwork, Bombieri [8], Adolphson-Sperber [1, 2], the second author [26, 27], and Blache [7] for  $\psi$  of order  $p$ . More recently initial part of the theory was extended to all  $\psi$  by Liu-Wei [20] and Liu [19].

The  $p$ -adic theory of the above exponential sum for  $n = 1$  and  $\psi$  of order  $p$  has a long history and has been studied extensively in the literature. For instance, in the simplest case that  $f(x) = x^d$ , the exponential sum was studied by Gauss, see Berndt-Evans [3] for a comprehensive survey. By the Hasse-Davenport relation for Gauss sums, the  $L$ -function is a polynomial whose zeros are given by roots of Gauss sums. Thus, the slopes of the  $L$ -function are completely determined by the Stickelberger theorem for Gauss sums. The roots of the  $L$ -function have explicit  $p$ -adic formulas in terms of  $p$ -adic  $\Gamma$ -function via the Gross-Koblitz formula [13]. These ideas can be extended to treat the so-called diagonal  $f$  case for general  $n$ , see Wan [27]. These elementary cases have been used as building bricks to study the deeper non-diagonal  $f(x)$  via various decomposition theorems, which are the main ideas of Wan [26, 27]. In the case  $n = 1$  and  $\psi$  of order  $p$ , further progresses about the slopes of the  $L$ -function were made in Zhu [32, 33], Blache and Ferard [5], and Liu [21].

**1.2.  $T$ -adic exponential sums.** We now define the  $T$ -adic exponential sum, state our main results, and put forward some new questions.

**Definition 1.2.** *For a positive integer  $k$ , the  $T$ -adic exponential sum of  $f$  over  $\mathbb{F}_{q^k}$  is the sum:*

$$S_f(k, T) = \sum_{x \in \mu_{q^k-1}^n} (1 + T)^{\text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} \in \mathbb{Z}_p[[T]].$$

*The  $T$ -adic  $L$ -function of  $f$  over  $\mathbb{F}_q$  is the generating function*

$$L_f(s, T) = L_f(s, T; \mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_f(k, T) \frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

The  $T$ -adic exponential sum interpolates classical exponential sums of  $p^m$ -order over finite fields for all positive integers  $m$ . In fact, we have

$$S_f(k, \pi_\psi) = S_{f,\psi}(k).$$

Similarly, one can recover the classical L-function of the  $p^m$ -order exponential sum from the  $T$ -adic  $L$ -function by the formula

$$L_f(s, \pi_\psi) = L_{f,\psi}(s).$$

We view  $L_f(s, T)$  as a power series in the single variable  $s$  with coefficients in the complete discrete valuation ring  $\mathbb{Q}_p[[T]]$  with uniformizer  $T$ .

**Definition 1.3.** *The  $T$ -adic characteristic function of  $f$  over  $\mathbb{F}_q$ , or  $C$ -function of  $f$  for short, is the generating function*

$$C_f(s, T) = \exp\left(\sum_{k=1}^{\infty} -(q^k - 1)^{-n} S_f(k, T) \frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

The  $C$ -function  $C_f(s, T)$  and the  $L$ -function  $L_f(s, T)$  determine each other. They are related by

$$L_f(s, T) = \prod_{i=0}^n C_f(q^i s, T)^{(-1)^{n-i-1} \binom{n}{i}},$$

and

$$C_f(s, T)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_f(q^j s, T)^{\binom{n+j-1}{j}}.$$

In §4, we prove

**Theorem 1.4** (analytic continuation). *The  $C$ -function  $C_f(s, T)$  is  $T$ -adic entire in  $s$ . As a consequence, the  $L$ -function  $L_f(s, T)$  is  $T$ -adic meromorphic in  $s$ .*

The above theorem tells that the  $C$ -function behaves  $T$ -adically better than the  $L$ -function. In fact, in the  $T$ -adic setting, the  $C$ -function is a more natural object than the  $L$ -function. Thus, we shall focus more on the  $C$ -function.

Knowing the analytic continuation of  $C_f(s, T)$ , we are then interested in the location of its zeros. More precisely, we would like to determine the  $T$ -adic Newton polygon of this entire function  $C_f(s, T)$ . This is expected to be a complicated problem in general. It is open even in the simplest case  $n = 1$  and  $f(x) = x^d$  is a monomial if  $p \not\equiv 1 \pmod{d}$ . What we can do is to give an explicit combinatorial lower bound depending only on  $q$  and  $\Delta$ , called the  $q$ -Hodge bound  $\text{HP}_q(\Delta)$ . This polygon will be described in detail in §3.

Let  $\text{NP}_T(f)$  denote the  $T$ -adic Newton polygon of the  $C$ -function  $C_f(s, T)$ . In §5, we prove

**Theorem 1.5** (Hodge bound). *We have*

$$\mathrm{NP}_T(f) \geq \mathrm{HP}_q(\Delta).$$

This theorem shall give several new results on classical exponential sums, as we shall see in §2. In particular, this extends, in one stroke, all known ordinariness results for  $\psi$  of order  $p$  to all  $\psi$  of any  $p$ -power order. It demonstrates the significance of the  $T$ -adic L-function. It also gives rise to the following definition.

**Definition 1.6.** *The Laurent polynomial  $f$  is called  $T$ -adically ordinary if  $\mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta)$ .*

We shall show that the classical notion of ordinariness implies  $T$ -adic ordinariness. But it is possible that a non-ordinary  $f$  is  $T$ -adically ordinary. Thus, it remains of interest to study exactly when  $f$  is  $T$ -adically ordinary. For this purpose, in §6, we extend the facial decomposition theorem in Wan [26] to the  $T$ -adic case. Let  $\Delta$  be the convex closure in  $\mathbb{R}^n$  of the origin and the exponents of the non-zero monomials in the Laurent polynomial  $f(x)$ . For any closed face  $\sigma$  of  $\Delta$ , we let  $f_\sigma$  denote the sum of monomials of  $f$  whose exponent vectors lie in  $\sigma$ .

**Theorem 1.7** ( $T$ -adic facial decomposition). *The Laurent polynomial  $f$  is  $T$ -adically ordinary if and only if for every closed face  $\sigma$  of  $\Delta$  of codimension 1 not containing the origin, the restriction  $f_\sigma$  is  $T$ -adically ordinary.*

In §7, we briefly discuss the variation of the  $C$ -function  $C_f(s, T)$  and its Newton polygon when the reduction of  $f$  moves in an algebraic family over a finite field. The main questions are the generic ordinariness, generic Newton polygon, the analogue of the Adolphson-Sperber conjecture [1], Wan's limiting conjecture [27], Dwork's unit root conjecture [10] in the  $T$ -adic and  $\pi_\psi$ -adic case. We shall give an overview about what can be proved and what is unknown, including a number of conjectures. Basically, a lot can be proved in the ordinary case, and a lot remain to be proved in the non-ordinary case.

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## 2. APPLICATIONS

In this section we give several applications of the  $T$ -adic exponential sum to classical exponential sums.

**Theorem 2.1** (integrality theorem). *We have*

$$L_f(s, T) \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

and

$$C_f(s, T) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

*Proof.* Let  $|\mathbb{G}_m^n|$  be the set of closed points of  $\mathbb{G}_m^n$  over  $\mathbb{F}_q$ , and  $a \mapsto \hat{a}$  the Teichmüller lifting. It is easy to check that the  $T$ -adic L-function has the Euler product expansion

$$L_f(s, T) = \prod_{x \in |\mathbb{G}_m^n|} \frac{1}{(1 - (1 + T)^{\text{Tr}_{\mathbb{Q}_q^{\deg(x)}/\mathbb{Q}_p}(f(\hat{x}))} s^{\deg(x)})} \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where  $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ . The theorem now follows.  $\square$

The above proof shows that the L-function  $L_f(s, T)$  is the L-function  $L(s, \rho_f)$  of the following continuous  $(p, T)$ -adic representation of the arithmetic fundamental group:

$$\rho_f : \pi_1^{\text{arith}}(\mathbb{G}_m^n/\mathbb{F}_q) \longrightarrow \text{GL}_1(\mathbb{Z}_p[[T]]),$$

defined by

$$\rho_f(\text{Frob}_x) = (1 + T)^{\text{Tr}_{\mathbb{Q}_q^{\deg(x)}/\mathbb{Q}_p}(f(\hat{x}))}.$$

The rank one representation  $\rho_f$  is transcendental in nature. Its L-function  $L(s, \rho_f)$  seems to be beyond the reach of  $\ell$ -adic cohomology, where  $\ell$  is a prime different from  $p$ . However, the specialization of  $\rho_f$  at the special point  $T = \pi_\psi$  is a character of finite order. Thus, the specialization

$$L(s, \rho_f)|_{T=\pi_\psi} = L_{f,\psi}(s)$$

can indeed be studied using Grothendieck's  $\ell$ -adic trace formula [14]. This gives another proof that the L-function  $L_{f,\psi}(s)$  is a rational function in  $s$ . But the  $T$ -adic L-function  $L_f(s, T)$  itself is certainly out of the reach of  $\ell$ -adic cohomology as it is truly transcendental.

Let  $\text{NP}_T(f)$  denote the  $T$ -adic Newton polygon of the  $C$ -function  $C_f(s, T)$ , and let  $\text{NP}_{\pi_\psi}(f)$  denote the  $\pi_\psi$ -adic Newton polygon of the  $C$ -function  $C_f(s, \pi_\psi)$ . The integrality of  $C_f(s, T)$  immediately gives the following theorem.

**Theorem 2.2** (rigidity bound). *If  $\psi$  is non-trivial, then*

$$\text{NP}_{\pi_\psi}(f) \geq \text{NP}_T(f).$$

*Proof.* Obvious.  $\square$

A natural question is to ask when  $\text{NP}_{\pi_\psi}(f)$  coincides with its rigidity bound.

**Theorem 2.3** (transfer theorem). *If  $\text{NP}_{\pi_\psi}(f) = \text{NP}_T(f)$  holds for one non-trivial  $\psi$ , then it holds for all non-trivial  $\psi$ .*

*Proof.* By the integrality of  $C_f(s, T)$ , the  $T$ -adic Newton polygon of  $C_f(s, T)$  coincides with the  $\pi_\psi$ -adic Newton polygon of  $C_f(s, \pi_\psi)$  if and only if for every vertex  $(i, e)$  of the  $T$ -adic Newton polygon of  $C_f(s, T)$ , the coefficients of  $s^i$  in  $C_f(s, T)$  differs from  $T^e$  by a unit in  $\mathbb{Z}_p[[T]]^\times$ . It follows that if the coincidence happens for one non-trivial  $\psi$ , it happens for all non-trivial  $\psi$ . The theorem is proved.  $\square$

**Definition 2.4.** We call  $f$  rigid if  $\text{NP}_{\pi_\psi}(f) = \text{NP}_T(f)$  for one (and hence for all) non-trivial  $\psi$ .

In [22], cooperating with his students, the first author showed that  $f$  is generically rigid if  $n = 1$  and  $p$  is sufficiently large. So the rigid bound is the best possible bound. In contrast, the weaker Hodge bound  $\text{HP}_q(\Delta)$  is only best possible if  $p \equiv 1 \pmod{d}$ , where  $d$  is the degree of  $f$ .

We now pause to describe the relationship between the Newton polygons of  $C_f(s, \pi_\psi)$  and  $L_{f,\psi}(s)^{(-1)^{n-1}}$ . We need the following definitions.

**Definition 2.5.** A convex polygon with initial point  $(0, 0)$  is called algebraic if it is the graph of a  $\mathbb{Q}$ -valued function defined on  $\mathbb{N}$  or on an interval of  $\mathbb{N}$ , and its slopes are of finite multiplicity and of bounded denominator.

**Definition 2.6.** For an algebraic polygon with slopes  $\{\lambda_i\}$ , we define its slope series to be  $\sum_i t^{\lambda_i}$ .

It is clear that an algebraic polygon is uniquely determined by its slope series. So the slope series embeds the set of algebraic polygons into the ring  $\varinjlim_d \mathbb{Z}[[t^{\frac{1}{d}}]]$ . The image is  $\varinjlim_d \mathbb{N}[[t^{\frac{1}{d}}]]$ . It is closed under addition and multiplication. Therefore one can define an addition and a multiplication on the set of algebraic polygons.

**Lemma 2.7.** Suppose that  $f$  is non-degenerate. Then the  $q$ -adic Newton polygon of  $C_f(s, \pi_\psi; \mathbb{F}_q)$  is the product of the  $q$ -adic Newton polygon of  $L_{f,\psi}(s; \mathbb{F}_q)^{(-1)^{n-1}}$  and the algebraic polygon  $\frac{1}{(1-t)^n}$ .

*Proof.* Note that the  $C$ -value  $C_f(s, \pi_\psi)$  and the  $L$ -function  $L_{f,\psi}(s)$  determine each other. They are related by

$$L_{f,\psi}(s) = \prod_{i=0}^n C_f(q^i s, \pi_\psi)^{(-1)^{n-i-1} \binom{n}{i}},$$

and

$$C_f(s, \pi_\psi)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_{f,\psi}(q^j s)^{\binom{n+j-1}{j}}.$$

Suppose that

$$L_{f,\psi}(s)^{(-1)^{n-1}} = \prod_{i=1}^d (1 - \alpha_i s).$$

Then

$$C_f(s, \pi_\psi) = \prod_{j=0}^{\infty} \prod_{i=1}^d (1 - \alpha_i q^j s)^{\binom{n+j-1}{j}}.$$

Let  $\lambda_i$  be the  $q$ -adic order of  $\alpha_i$ . Then the  $q$ -adic order of  $\alpha_i q^j$  is  $\lambda_i + j$ . So the slope series of the  $q$ -adic Newton polygon of  $L_{f,\psi}(s)^{(-1)^{n-1}}$  is

$$S(t) = \sum_{i=1}^d t^{\lambda_i},$$

and the slope series of the  $q$ -adic Newton polygon of  $C_f(s, \pi_\psi)$  is

$$\sum_{j=0}^{+\infty} \sum_{i=0}^d \binom{n+j-1}{j} t^{\lambda_i+j} = \frac{1}{(1-t)^n} S(t).$$

The lemma now follows.  $\square$

We combine the rigidity bound and the Hodge bound to give the following theorem.

**Theorem 2.8.** *If  $\psi$  is non-trivial, then*

$$\text{NP}_{\pi_\psi}(f) \geq \text{NP}_T(f) \geq \text{HP}_q(\Delta).$$

*Proof.* Obvious.  $\square$

If we drop the middle term, we arrive at the Hodge bound

$$\text{NP}_{\pi_\psi}(f) \geq \text{HP}_q(\Delta)$$

of Adolphson-Sperber [2] and Liu-Wei [20].

**Theorem 2.9.** *If  $\text{NP}_{\pi_\psi}(f) = \text{HP}_q(\Delta)$  holds for one non-trivial  $\psi$ , then  $f$  is rigid,  $T$ -adically ordinary, and the equality holds for all non-trivial  $\psi$ .*

*Proof.* Suppose that  $\text{NP}_{\pi_{\psi_0}}(f) = \text{HP}_q(\Delta)$  for a non-trivial  $\psi_0$ . Then, by the last theorem, we have

$$\text{NP}_{\pi_{\psi_0}}(f) = \text{NP}_T(f) = \text{HP}_q(\Delta).$$

So  $f$  is rigid and  $T$ -adically ordinary, and

$$\text{NP}_{\pi_\psi}(f) = \text{NP}_T(f) = \text{HP}_q(\Delta)$$

holds for all nontrivial  $\psi$ . The theorem is proved.  $\square$

**Definition 2.10.** *We call  $f$  ordinary if  $\text{NP}_{\pi_\psi}(f) = \text{HP}_q(\Delta)$  holds for one (and hence for all) non-trivial  $\psi$ .*

The notion of ordinariness now carries much more information than what we had known. From this, we see that the  $T$ -adic exponential sum provides a new framework to study all  $p^m$ -power order exponential sums simultaneously. Instead of the usual way of extending the methods for  $\psi$  of order  $p$  to the case of higher order, the  $T$ -adic exponential sum has the novel feature that it can sometimes transfer a known result for one non-trivial  $\psi$  to all non-trivial  $\psi$ . This philosophy is carried out further in the paper [22].

**Example 2.1.** *Let*

$$f(x) = x_1 + x_2 + \cdots + x_n + \frac{\alpha}{x_1 x_2 \cdots x_n}, \quad \alpha \in \mu_{q-1}.$$

*Then, by the result of Sperber [25] and our new information on ordinariness, we have*

$$\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$$

*for all non-trivial  $\psi$ .*

### 3. THE $q$ -HODGE POLYGON

In this section, we describe explicitly the  $q$ -Hodge polygon mentioned in the introduction. Recall that  $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$  is a Laurent polynomial in  $n$  variables of the form

$$f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u, \quad a_u \in \mathbb{Z}_q, \quad a_u^q = a_u.$$

We stress that the non-zero coefficients of  $f(x)$  are roots of unity in  $\mathbb{Z}_q$ , thus correspond in a unique way to Teichmüller liftings of elements of the finite field  $\mathbb{F}_q$ . If the coefficients of  $f(x)$  are arbitrary elements in  $\mathbb{Z}_q$ , much of the theory still holds, but it is more complicated to describe the results. We have made the simplifying assumption that the non-zero coefficients are always roots of unity in this paper.

Let  $\Delta$  be the convex polyhedron in  $\mathbb{R}^n$  associated to  $f$ , which is generated by the origin and the exponent vectors of the non-zero monomials of  $f$ . Let  $C(\Delta)$  be the cone in  $\mathbb{R}^n$  generated by  $\Delta$ . Define the degree function  $u \mapsto \deg(u)$  on  $C(\Delta)$  such that  $\deg(u) = 1$  when  $u$  lies on a codimensional 1 face of  $\Delta$  that does not contain the origin, and such that

$$\deg(ru) = r \deg(u), \quad r \in \mathbb{R}_{\geq 0}, \quad u \in C(\Delta).$$

We call it the degree function associated to  $\Delta$ . We have  $\deg(u + v) \leq \deg(u) + \deg(v)$  if  $u, v \in C(\Delta)$ , and the equality holds if and only if  $u$  and  $v$  are co-facial. In other words, the number

$$c(u, v) := \deg(u) + \deg(v) - \deg(u + v)$$

is 0 if  $u, v \in C(\Delta)$  are co-facial, and is positive otherwise. We call that number  $c(u, v)$  the co-facial defect of  $u$  and  $v$ . Let

$$M(\Delta) := C(\Delta) \cap \mathbb{Z}^n$$

be the set of lattice points in the cone  $C(\Delta)$ . Let  $D$  be the denominator of the degree function, which is the smallest positive integer such that

$$\deg M(\Delta) \subset \frac{1}{D} \mathbb{Z}.$$

For every natural number  $k$ , we define

$$W(k) := W_\Delta(k) = \#\{u \in M(\Delta) \mid \deg(u) = k/D\}$$



to be the number of lattice points of degree  $\frac{k}{D}$  in  $M(\Delta)$ . For prime power  $q = p^a$ , the  $q$ -Hodge polygon of  $f$  is the polygon with vertices  $(0, 0)$  and

$$\left( \sum_{j=0}^i W(j), a(p-1) \sum_{j=0}^i \frac{j}{D} W(j) \right), \quad i = 0, 1, \dots.$$

It is also called the  $q$ -Hodge polygon of  $\Delta$  and denoted by  $\text{HP}_q(\Delta)$ . It depends only on  $q$  and  $\Delta$ . It has a side of slope  $a(p-1)\frac{j}{D}$  with horizontal length  $W(j)$  for each non-negative integer  $j$ .

#### 4. ANALYTIC CONTINUATION

In this section, we prove the  $T$ -adic analytic continuation of the C-function  $C_f(s, T)$ . The idea is to employ Dwork's trace formula in the  $T$ -adic case.

Note that the Galois group  $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$  is cyclic of order  $a = \log_p q$ . There is an element in the Galois group whose restriction to  $\mu_{q-1}$  is the  $p$ -power morphism. It is of order  $a$ , and is called the Frobenius element. We denote that element by  $\sigma$ .

We define a new variable  $\pi$  by the relation  $E(\pi) = 1 + T$ , where

$$E(\pi) = \exp\left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i}\right) \in 1 + \pi\mathbb{Z}_p[[\pi]]$$

is the Artin-Hasse exponential series. Thus,  $\pi$  and  $T$  are two different uniformizers of the  $T$ -adic local ring  $\mathbb{Q}_p[[T]]$ . It is clear that for  $\alpha \in \mathbb{Z}_q$ , we have

$$E(\pi\alpha) \in 1 + \pi\mathbb{Z}_q[[\pi]],$$

and for  $\beta \in \mathbb{Z}_p$ , we have

$$E(\pi)^\beta \in 1 + \pi\mathbb{Z}_p[[\pi]].$$

The Galois group  $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$  can act on  $\mathbb{Z}_q[[\pi]]$  but keeping  $\pi$  fixed. The Artin-Hasse exponential series has a kind of commutativity expressed as the following lemma.

**Lemma 4.1** (Commutativity). *We have the following commutative diagram*

$$\begin{array}{ccc} \mu_{q-1} & \xrightarrow{E(\pi \cdot)} & \mathbb{Z}_q[[\pi]] \\ \text{Tr} \downarrow & & \downarrow \text{Norm} \\ \mu_{p-1} & \xrightarrow{E(\pi) \cdot} & \mathbb{Z}_p[[\pi]]. \end{array}$$

That is, if  $x \in \mu_{q-1}$ , then

$$E(\pi)^{x+x^p+\dots+x^{p^{a-1}}} = E(\pi x)E(\pi x^p) \cdots E(\pi x^{p^{a-1}}).$$

*Proof.* Since for  $x \in \mu_{q-1}$ ,

$$\sum_{j=0}^{a-1} x^{p^j} = \sum_{j=0}^{a-1} x^{p^{j+i}},$$

we have

$$E(\pi)^{x+x^p+\dots+x^{p^{a-1}}} = \exp\left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} \sum_{j=0}^{a-1} x^{p^{j+i}}\right) = E(\pi x)E(\pi x^p) \cdots E(\pi x^{p^{a-1}}).$$

The lemma is proved.  $\square$

**Definition 4.2.** Let  $\pi^{1/D}$  be a fixed  $D$ -th root of  $\pi$ . Define

$$L(\Delta) = \left\{ \sum_{u \in M(\Delta)} b_u \pi^{\deg(u)} x^u : b_u \in \mathbb{Z}_q[[\pi^{1/D}]] \right\},$$

and

$$B = \left\{ \sum_{u \in M(\Delta)} b_u \pi^{\deg(u)} x^u \in L(\Delta), \text{ord}_T(b_u) \rightarrow +\infty \text{ if } \deg(u) \rightarrow +\infty \right\}.$$

The spaces  $L(\Delta)$  and  $B$  are  $T$ -adic Banach algebras over the ring  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . The monomials  $\pi^{\deg(u)} x^u$  ( $u \in M(\Delta)$ ) form an orthonormal basis (resp., a formal basis) of  $B$  (resp.,  $L(\Delta)$ ). The algebra  $B$  is contained in the larger Banach algebra  $L(\Delta)$ . If  $u \in \Delta$ , it is clear that  $E(\pi x^u) \in L(\Delta)$ . Write

$$E_f(x) := \prod_{a_u \neq 0} E(\pi a_u x^u), \text{ if } f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u.$$

This is an element of  $L(\Delta)$  since  $L(\Delta)$  is a ring.

The Galois group  $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$  can act on  $L(\Delta)$  but keeping  $\pi^{1/D}$  as well as the variables  $x_i$ 's fixed. From the commutativity of the Artin-Hasse exponential series, one can infer the following lemma.

**Lemma 4.3** (Dwork's splitting lemma). *If  $x \in \mu_{q^k-1}$ , then*

$$E(\pi)^{\text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}),$$

where  $a$  is the order of  $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ .

*Proof.* We have

$$\begin{aligned} E(\pi)^{\text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} &= \prod_{a_u \neq 0} E(\pi)^{\text{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(a_u x^u)} \\ &= \prod_{a_u \neq 0} \prod_{i=0}^{ak-1} E(\pi (a_u x^u)^{p^i}) = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}). \end{aligned}$$

The lemma is proved.  $\square$

**Definition 4.4.** We define a map

$$\psi_p : L(\Delta) \rightarrow L(\Delta), \quad \sum_{u \in M(\Delta)} b_u x^u \mapsto \sum_{u \in M(\Delta)} b_{pu} x^u.$$

It is clear that the composition map  $\psi_p \circ E_f$  sends  $B$  to  $B$ .

**Lemma 4.5.** *Write*

$$E_f(x) = \sum_{u \in M(\Delta)} \alpha_u(f) \pi^{\deg(u)} x^u.$$

$$\begin{aligned} & \text{Then, } \psi_p \circ E_f(\pi^{\deg(u)} x^u) \\ &= \sum_{w \in M(\Delta)} \alpha_{pw-u}(f) \pi^{c(pw-u, u)} \pi^{(p-1)\deg(w)} \pi^{\deg(w)} x^w, \quad u \in M(\Delta), \end{aligned}$$

where  $c(pw - u, u)$  is the co-facial defect of  $pw - u$  and  $u$ .

*Proof.* This follows directly from the definition of  $\psi_p$  and  $E_f(x)$ .  $\square$

**Definition 4.6.** *Define*

$$\psi := \sigma^{-1} \circ \psi_p \circ E_f : B \longrightarrow B,$$

and its  $a$ -th iterate

$$\psi^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}).$$

Note that  $\psi$  is linear over  $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$ , but semi-linear over  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . On the other hand,  $\psi^a$  is linear over  $\mathbb{Z}_q[[\pi^{1/D}]]$ . By the last lemma,  $\psi^a$  is completely continuous in the sense of Serre [24].

**Theorem 4.7** (Dwork's trace formula). *For every positive integer  $k$ ,*

$$(q^k - 1)^{-n} S_f(k, T) = \text{Tr}_{B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]}(\psi^{ak}).$$

*Proof.* Let  $g(x) \in B$ . We have

$$\psi^{ak}(g) = \psi_p^{ak}(g \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i})).$$

Write

$$\prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_u x^u.$$

One computes that

$$\psi^{ak}(\pi^{\deg(v)} x^v) = \sum_{u \in M(\Delta)} \beta_{q^k u - v} \pi^{\deg(v)} x^u.$$

Thus,

$$\text{Tr}(\psi^{ak} | B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}.$$

But, by Dwork's splitting lemma, we have

$$(q^k - 1)^{-n} S_f(k, T) = (q^k - 1)^{-n} \sum_{x \in \mu_{q^k-1}^n} \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}.$$

The theorem now follows.  $\square$

**Theorem 4.8** (Analytic trace formula). *We have*

$$C_f(s, T) = \det(1 - \psi^a s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]).$$

*In particular, the  $T$ -adic  $C$ -function  $C_f(s, T)$  is  $T$ -adic analytic in  $s$ .*

*Proof.* It follows from the last theorem and the well known identity

$$\det(1 - \psi^a s) = \exp\left(-\sum_{k=1}^{\infty} \text{Tr}(\psi^{ak}) \frac{s^k}{k}\right).$$

□

This theorem gives another proof that the coefficients of  $C_f(s, T)$  and  $L_f(s, T)$  as power series in  $s$  are  $T$ -adically integral.

**Corollary 4.9.** *For each non-trivial  $\psi$ , the  $C$ -value  $C_f(s, \pi_\psi)$  is  $p$ -adic entire in  $s$  and the  $L$ -function  $L_{f, \psi}(s)$  is rational in  $s$ .*

*Proof.* Obvious. □

## 5. THE HODGE BOUND

The analytic trace formula in the previous section reduces the study of  $C_f(s, T)$  to the study of the operator  $\psi^a$ . We consider  $\psi$  first. Note that  $\psi$  operates on  $B$  and is linear over  $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$ .

**Theorem 5.1.** *The  $T$ -adic Newton polygon of  $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$  lies above the polygon with vertices  $(0, 0)$  and*

$$(a \sum_{k=0}^i W(k), a(p-1) \sum_{k=0}^i \frac{k}{D} W(k)), \quad i = 0, 1, \dots$$

*Proof.* Let  $\xi_1, \xi_2, \dots, \xi_a$  be a normal basis of  $\mathbb{Q}_q$  over  $\mathbb{Q}_p$ . Write

$$(\xi_j \alpha_{pw-u}(f))^{\sigma^{-1}} = \sum_{i=0}^{a-1} \alpha_{(i,w),(j,u)}(f) \xi_i, \quad \alpha_{(i,w),(j,u)}(f) \in \mathbb{Z}_p[[\pi^{1/D}]].$$

Then  $\psi(\xi_j \pi^{\deg(u)} x^u)$

$$= \sum_{i=0}^{a-1} \sum_{w \in M(\Delta)} \alpha_{(i,w),(j,u)}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg(w)} \xi_i \pi^{\deg(w)} x^w.$$

That is, the matrix of  $\psi$  over  $\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]$  with respect to the orthonormal basis  $\{\xi_j \pi^{\deg(u)} x^u\}_{0 \leq j < a, u \in M(\Delta)}$  is

$$A = (\alpha_{(i,w),(j,u)}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg(w)})_{(i,w),(j,u)}.$$

So, the  $T$ -adic Newton polygon of  $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$  lies above the polygon with vertices  $(0, 0)$  and

$$(a \sum_{k=0}^i W(k), a(p-1) \sum_{k=0}^i \frac{k}{D} W(k)) \quad (i = 0, 1, \dots).$$

Theorem 5.1 is proved.  $\square$

We are now ready to prove the Hodge bound for the Newton polygon.

**Theorem 5.2.** *We have*

$$\text{NP}_T(f) \geq \text{HP}_q(\Delta).$$

*Proof.* By the above theorem, it suffices to prove that the  $T$ -adic Newton polygon of  $\det(1 - \psi^a s^a \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])$  coincides with that of  $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$ . Note that

$$\det(1 - \psi^a s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]) = \text{Norm}(\det(1 - \psi^a s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])),$$

where the norm map is the norm from  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$  to  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . The theorem now follows from the equality

$$\prod_{\zeta^a=1} \det(1 - \psi \zeta s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]) = \det(1 - \psi^a s^a \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]).$$

$\square$

## 6. FACIAL DECOMPOSITION

In this section, we extend the facial decomposition theorem in [26]. Recall that the operator  $\psi = \sigma^{-1} \circ (\psi_p \circ E_f)$  is only semi-linear over  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . But its second factor  $\psi_p \circ E_f$  is clearly linear and so  $\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])$  is well defined. We begin with the following theorem.

**Theorem 6.1.** *The  $T$ -adic Newton polygon of  $f$  coincides with  $\text{HP}_q(\Delta)$  if and only if the  $T$ -adic Newton polygon of  $\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])$  coincides with the polygon with vertices  $(0, 0)$  and*

$$\left( \sum_{k=0}^i W(k), (p-1) \sum_{k=0}^i \frac{k}{D} W(k) \right), \quad i = 0, 1, \dots$$

*Proof.* In the proof of Theorem 5.2, we showed that the  $T$ -adic Newton polygon of  $C_f(s^a, T)$  coincides with that of  $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$ . Note that

$$\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]]) = \text{Norm}(\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{D}}]])),$$

where the norm map is the norm from  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$  to  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . The theorem is equivalent to the statement that the  $T$ -adic Newton polygon of  $\det(1 - \psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$  coincides with the polygon with vertices  $(0, 0)$  and

$$\left( \sum_{k=0}^i aW(k), a(p-1) \sum_{k=0}^i \frac{k}{D} W(k) \right), \quad i = 0, 1, \dots$$

if and only if the  $T$ -adic Newton polygon of  $\det(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{D}}]])$  does. Therefore it suffices to show that the determinant of the matrix

$$(\alpha_{(i,w),(j,u)}(f) \pi^{c(pw-u,u)})_{0 \leq i,j < a, \deg(w), \deg(u) \leq \frac{k}{D}}$$

is not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$  if and only if the determinant of the matrix

$$(\alpha_{pw-u}(f)\pi^{c(pw-u,u)})_{\deg(w), \deg(u) \leq \frac{k}{D}}$$

is not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . The theorem now follows from the fact that the latter determinant is the norm of the former from  $\mathbb{Q}_q[[\pi^{\frac{1}{D}}]]$  to  $\mathbb{Q}_p[[\pi^{\frac{1}{D}}]]$  up to a sign.  $\square$

We now define the open facial decomposition  $F(\Delta)$ . It is the decomposition of  $C(\Delta)$  into a disjoint union of relatively open cones generated by the relatively open faces of  $\Delta$  whose closure does not contain the origin. Note that every relatively open cone generated by co-facial vectors in  $C(\Delta)$  is contained in a unique element of  $F(\Delta)$ .

**Lemma 6.2.** *Let  $\sigma \in F(\Delta)$ , and  $u \in \sigma$ . Then  $\alpha_u(f_{\bar{\sigma}}) \equiv \alpha_u(f) \pmod{\pi^{1/D}}$ , where  $f_{\bar{\sigma}}$  is the sum of monomials of  $f$  whose exponent vectors lie in the closure  $\bar{\sigma}$  of  $\sigma$ .*

*Proof.* Let  $v_1, \dots, v_j$  be exponent vectors of monomials of  $f$  such that  $a_1 v_1 + \dots + a_j v_j = u$  with  $a_1 > 0, \dots, a_j > 0$ . It suffices to show that either  $v_1, \dots, v_j$  lie in the closure of  $\sigma$ , or their contribution to  $\alpha_u(f)$  is  $\equiv 0 \pmod{\pi^{1/D}}$ . Suppose that their contribution to  $\alpha_u(f)$  is  $\not\equiv 0 \pmod{\pi^{1/D}}$ . Then  $v_1, \dots, v_j$  must be co-facial. So the interior of the cone generated by those vectors is contained in a unique element of  $F(\Delta)$ . As that interior has a common point  $u$  with  $\sigma$ , it must be  $\sigma$ . It follows that  $v_1, \dots, v_j$  lie in the closure of  $\sigma$ . The lemma is proved.  $\square$

**Lemma 6.3.** *Let  $\sigma, \tau \in F(\Delta)$  be distinct. Let  $w \in \sigma$ , and  $u \in \tau$ . Suppose that the dimension of  $\sigma$  is no greater than that of  $\tau$ . Then  $pw - u$  and  $u$  are not co-facial, i.e.,  $c(pw - u, u) > 0$ .*

*Proof.* Suppose that  $pw - u$  and  $u$  are co-facial. Then the interior of the cone generated by  $pw - u$  and  $u$  is contained in a unique element of  $F(\Delta)$ . As that interior has a common point  $w$  with  $\sigma$ , it must be  $\sigma$ . It follows that  $u$  lies in the closure of  $\sigma$ . As  $\sigma$  and  $\tau$  are distinct,  $u$  lies in the boundary of  $\sigma$ . This implies that the dimension of  $\tau$  is less than that of  $\sigma$ , which is a contradiction. Therefore  $pw - u$  and  $u$  are not co-facial. The lemma is proved.  $\square$

For  $\sigma \in F(\Delta)$ , we define

$$M(\sigma) = M(\Delta) \cap \sigma = \mathbb{Z}^n \cap \sigma$$

be the set of lattice points in the cone  $\sigma$ .

**Theorem 6.4** (Open facial decomposition). *The  $T$ -adic Newton polygon of  $f$  coincides with  $\text{HP}_q(\Delta)$  if and only if for every  $\sigma \in F(\Delta)$ , the determinants of the matrices*

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u \in M(\sigma), \deg(w), \deg(u) \leq \frac{k}{D}}, \quad k = 0, 1, \dots$$

*are not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ , where  $\bar{\sigma}$  is the closure of  $\sigma$ .*

*Proof.* By Theorem 6.1, the  $T$ -adic Newton polygon of  $C_f(s, T)$  coincides with the  $q$ -Hodge polygon of  $f$  if and only if the determinants of the matrices

$$A^{(k)} = \{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w,u \in M(\Delta), \deg(w), \deg(u) \leq \frac{k}{D}}, \quad k = 0, 1, \dots$$

are not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . Write

$$A_{\sigma,\tau}^{(k)} = \{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w \in M(\sigma), u \in M(\tau), \deg(w), \deg(u) \leq \frac{k}{D}}.$$

The facial decomposition shows that  $A^{(k)}$  has the block form  $(A_{\sigma,\tau}^{(k)})_{\sigma,\tau \in F(\Delta)}$ . The last lemma shows that the block form modulo  $\pi^{\frac{1}{D}}$  is triangular if we order the cones in  $F(\Delta)$  in dimension-increasing order. It follows that  $\det A^{(k)}$  is not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$  if and only if for all  $\sigma \in F(\Delta)$ ,  $\det A_{\sigma,\sigma}^{(k)}$  is not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . By Lemma 6.2, modulo  $\pi^{\frac{1}{D}}$ ,  $A_{\sigma,\sigma}^{(k)}$  is congruent to the matrix

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u \in M(\sigma), \deg(w), \deg(u) \leq \frac{k}{D}}.$$

So  $\det A_{\sigma,\sigma}^{(k)}$  is not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$  if and only if the determinant of the matrix

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u \in M(\sigma), \deg(w), \deg(u) \leq \frac{k}{D}}$$

is not divisible by  $T$  in  $\mathbb{Z}_q[[\pi^{\frac{1}{D}}]]$ . The theorem is proved.  $\square$

The closed facial decomposition Theorem 1.7 follows from the open decomposition theorem and the fact that

$$F(\Delta) = \bigcup_{\sigma \in F(\Delta), \dim \sigma = \dim \Delta} F(\bar{\sigma}).$$

A similar  $\pi_\psi$ -adic facial decomposition theorem for  $C_f(s, \pi_\psi)$  can be proved in a similar way. Alternatively, it follows from the transfer theorem together with the  $\pi_\psi$ -adic facial decomposition in [26] for  $\psi$  of order  $p$ .

## 7. VARIATION OF C-FUNCTIONS IN A FAMILY

Fix an  $n$ -dimensional integral convex polytope  $\Delta$  in  $\mathbb{R}^n$  containing the origin. For each prime  $p$ , let  $P(\Delta, \mathbb{F}_p)$  denote the parameter space of all Laurent polynomials  $f(x)$  over  $\bar{\mathbb{F}}_p$  such that  $\Delta(f) = \Delta$ . This is a connected rational variety defined over  $\mathbb{F}_p$ . For each  $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$ , the Teichmüller lifting gives a Laurent polynomial  $\tilde{f}$  whose non-zero coefficients are roots of unity in  $\mathbb{Z}_q$ . The C-function  $C_{\tilde{f}}(s, T)$  is then defined and  $T$ -adically entire. For simplicity of notation, we shall just write  $C_f(s, T)$  for  $C_{\tilde{f}}(s, T)$ , similarly,  $L_f(s, T)$  for  $L_{\tilde{f}}(s, T)$ . Thus, our C-function and L-function are now defined for Laurent polynomials over finite fields, via the Teichmüller lifting. We would like to study how  $C_f(s, T)$  varies when  $f$  varies in the algebraic variety  $P(\Delta, \mathbb{F}_p)$ .

Recall that for a closed face  $\sigma \in \Delta$ ,  $f_\sigma$  denotes the restriction of  $f$  to  $\sigma$ . That is,  $f_\sigma$  is the sum of those non-zero monomials in  $f$  whose exponents are in  $\sigma$ .

**Definition 7.1.** A Laurent polynomial  $f \in P(\Delta, \mathbb{F}_p)$  is called non-degenerate if for every closed face  $\sigma$  of  $\Delta$  of arbitrary dimension which does not contain the origin, the system

$$\frac{\partial f_\sigma}{\partial x_1} = \cdots = \frac{\partial f_\sigma}{\partial x_n} = 0$$

has no common zeros with  $x_1 \cdots x_n \neq 0$  over the algebraic closure of  $\mathbb{F}_p$ .

The non-degenerate condition is a geometric condition which insures that the associated Dwork cohomology can be calculated. In particular, it implies that, if  $\psi$  is of order  $p^m$ , then the L-function  $L_{f,\psi}(s)^{(-1)^{n-1}}$  is a polynomial in  $s$  whose degree is precisely  $n! \text{Vol}(\Delta) p^{n(m-1)}$ , see [20]. As a consequence, we deduce

**Theorem 7.2.** Let  $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$ . Write

$$L_f(s, T)^{(-1)^{n-1}} = \sum_{k=0}^{\infty} L_{f,k}(T) s^k, \quad L_{f,k}(T) \in \mathbb{Z}_p[[T]].$$

Assume that  $f$  is non-degenerate. Then for every positive integer  $m$  and all positive integer  $k > n! \text{Vol}(\Delta) p^{n(m-1)}$ , we have the following congruence in  $\mathbb{Z}_p[[T]]$ :

$$L_{f,k}(T) \equiv 0 \left( \text{mod } \frac{(1+T)^{p^m} - 1}{T} \right).$$

*Proof.* Write

$$\frac{(1+T)^{p^m} - 1}{T} = \prod (T - \xi).$$

The non-degenerate assumption implies that

$$L_f(s, \xi)^{(-1)^{n-1}} = \sum_{j=0}^{\infty} L_{f,j}(\xi) s^j,$$

is a polynomials in  $s$  of degree  $\leq n! \text{Vol}(\Delta) p^{n(m-1)} < k$ . It follows that  $L_{f,k}(\xi) = 0$  for all  $\xi$ . That is,  $L_{f,k}(T)$  is divisible by  $(T - \xi)$  for  $\xi$ . The theorem now follows.  $\square$

**Definition 7.3.** Let  $N(\Delta, \mathbb{F}_p)$  denote the subset of all non-degenerate Laurent polynomials  $f \in P(\Delta, \mathbb{F}_p)$ .

The subset  $N(\Delta, \mathbb{F}_p)$  is Zariski open in  $P(\Delta, \mathbb{F}_p)$ . It can be empty for some pair  $(\Delta, \mathbb{F}_p)$ . But, for a given  $\Delta$ ,  $N(\Delta, \mathbb{F}_p)$  is Zariski open dense in  $P(\Delta, \mathbb{F}_p)$  for all primes  $p$  except for possibly finitely many primes depending on  $\Delta$ . It is an interesting and independent question to classify the primes  $p$  for which  $N(\Delta, \mathbb{F}_p)$  is non-empty. This is related to the GKZ discriminant [12]. For simplicity, we shall only consider non-degenerate  $f$  in the following.



**7.1. Generic ordinariness.** The first question is how often  $f$  is  $T$ -adically ordinary when  $f$  varies in the non-degenerate locus  $N(\Delta, \mathbb{F}_p)$ . Let  $U_p(\Delta, T)$  be the subset of  $f \in N(\Delta, \mathbb{F}_p)$  such that  $f$  is  $T$ -adically ordinary, and  $U_p(\Delta)$  the subset of  $f \in N(\Delta, \mathbb{F}_p)$  such that  $f$  is ordinary. One can prove

**Lemma 7.4.** *The set  $U_p(\Delta)$  is Zariski open in  $N(\Delta, \mathbb{F}_p)$ .*

One can ask if  $U_p(\Delta, T)$  is also Zariski open in  $N(\Delta, \mathbb{F}_p)$ . We do not know the answer.

Our question is for which  $p$ ,  $U_p(\Delta)$  and  $U_p(\Delta, T)$  are Zariski dense in  $N(\Delta, \mathbb{F}_p)$ . The rigidity bound as well as the Hodge bound imply that

$$U_p(\Delta) \subseteq U_p(\Delta, T).$$

It follows that if  $U_p(\Delta)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$ , then  $U_p(\Delta, T)$  is also Zariski dense in  $N(\Delta, \mathbb{F}_p)$ .

The Adolphson-Sperber conjecture [1] says that if  $p \equiv 1 \pmod{D}$ , then  $U_p(\Delta)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$ . This conjecture was proved to be true in [26] [27] if  $n \leq 3$ . In particular, this implies

**Theorem 7.5.** *If  $p \equiv 1 \pmod{D}$  and  $n \leq 3$ , then  $U_p(\Delta, T)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$ .*

For  $n \geq 4$ , it was shown in [26] [27] that there is an effectively computable positive integer  $D^*(\Delta)$  depending only on  $\Delta$  such that if  $p \equiv 1 \pmod{D^*(\Delta)}$ , then  $U_p(\Delta)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$ . Thus, we obtain

**Theorem 7.6.** *For each  $\Delta$ , there is an effectively computable positive integer  $D^*(\Delta)$  such that if  $p \equiv 1 \pmod{D^*(\Delta)}$ , then  $U_p(\Delta, T)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$ .*

The smallest possible  $D^*(\Delta)$  is rather subtle to compute in general, and it can be much larger than  $D$ . We now state a conjecture giving reasonably precise information on  $D^*(\Delta)$ .

**Definition 7.7.** *Let  $S(\Delta)$  be the monoid generated by the degree 1 lattice points in  $M(\Delta)$ , i.e., those lattice points on the codimension 1 faces of  $\Delta$  not containing the origin. Define the exponent of  $\Delta$  by*

$$I(\Delta) = \inf\{d \in \mathbb{Z}_{>0} \mid dM(\Delta) \subseteq S(\Delta)\}.$$

If  $u \in M(\Delta)$ , then the degree of  $Du$  will be integral but  $Du$  may not be a non-negative integral combination of degree 1 elements in  $M(\Delta)$  and thus  $DM(\Delta)$  may not be a subset of  $S(\Delta)$ . It is not hard to show that  $I(\Delta) \geq D$ . In general they are different but they are equal if  $n \leq 3$ . This explains why the Adolphson-Sperber conjecture is true if  $n \leq 3$  and it can be false if  $n \geq 4$ . The following conjecture is a modified form, and it is a consequence of Conjecture 9.1 in [26].

**Conjecture 7.8.** *If  $p \equiv 1 \pmod{I(\Delta)}$ , then  $U_p(\Delta)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$ . In particular,  $U_p(\Delta, T)$  is Zariski dense in  $N(\Delta, \mathbb{F}_p)$  for such  $p$ .*

By the facial decomposition theorem, in proving the above conjecture, it is sufficient to assume that  $\Delta$  has only one codimension 1 face not containing the origin.

**7.2. Generic Newton polygon.** In the case that  $U_p(\Delta, T)$  is empty, we expect the existence of a generic  $T$ -adic Newton polygon. For this purpose, we need to re-scale the uniformizer. For  $f \in N(\Delta, \mathbb{F}_p)(\mathbb{F}_{p^a})$ , the  $T^{a(p-1)}$ -adic Newton polygon of  $C_f(s, T; \mathbb{F}_{p^a})$  is independent of the choice of  $a$  for which  $f$  is defined over  $\mathbb{F}_{p^a}$ . We call them the absolute  $T$ -adic Newton polygon of  $f$ .

**Conjecture 7.9.** *There is a Zariski open dense subset  $G_p(\Delta, T)$  of  $N(\Delta, \mathbb{F}_p)$  such that the absolute  $T$ -adic Newton polygon of  $f$  is constant for all  $f \in G_p(\Delta, T)$ . Denote this common polygon by  $\text{GNP}_p(\Delta, T)$ , and call it the generic Newton polygon of  $(\Delta, T)$ .*

More generally, one expects that much of classical theory for finite rank  $F$ -crystals extends to a certain nuclear infinite rank setting. This includes the classical Dieudonne-Manin isogeny theorem, the Grothendieck specialization theorem, the Katz isogeny theorem [18]. All these are essentially understood in the ordinary infinite rank case, but open in the non-ordinary infinite rank case.

Similarly, for each non-trivial  $\psi$ , there is a Zariski open dense subset  $G_p(\Delta, \psi)$  of  $N(\Delta, \mathbb{F}_p)$  such that the  $\pi_\psi^{a(p-1)}$ -adic Newton polygon of the  $C$ -value  $C_f(s, \pi_\psi; \mathbb{F}_{p^a})$  is constant for all  $f \in G_p(\Delta, \psi)$ . Denote this common polygon by  $\text{GNP}_p(\Delta, \psi)$ , and call it the generic Newton polygon of  $(\Delta, \psi)$ . The existence of  $G_p(\Delta, \psi)$  can be proved. Since the non-degenerate assumption implies that the  $C$ -function  $C_f(s, \pi_\psi)$  is determined by a single finite rank  $F$ -crystal via a Dwork type cohomological formula for  $L_{f, \psi}(s)$ . In the  $T$ -adic case, we are not aware of any such finite rank reduction.

Clearly, we have the relation

$$\text{GNP}_p(\Delta, \psi) \geq \text{GNP}_p(\Delta, T).$$

**Conjecture 7.10.** *If  $p$  is sufficiently large, then*

$$\text{GNP}_p(\Delta, \psi) = \text{GNP}_p(\Delta, T).$$

This conjecture is proved in the case  $n = 1$  in [22].

Let  $\text{HP}(\Delta)$  denote the absolute Hodge polygon with vertices  $(0, 0)$  and

$$\left( \sum_{k=0}^i W(k), \sum_{k=0}^i \frac{k}{D} W(k) \right), \quad i = 0, 1, \dots$$

Note that  $\text{HP}(\Delta)$  depends only on  $\Delta$ , not on  $q$  any more. It is re-scaled from the  $q$ -Hodge polygon  $\text{HP}_q(\Delta)$ . Clearly, we have the relation

$$\text{GNP}_p(\Delta, \psi) \geq \text{GNP}_p(\Delta, T) \geq \text{HP}(\Delta).$$

Conjecture 7.8 says that if  $p \equiv 1 \pmod{I(\Delta)}$ , then both  $\text{GNP}_p(\Delta, \psi)$  and  $\text{GNP}_p(\Delta, T)$  are equal to  $\text{HP}(\Delta)$ . In general, the generic Newton polygon

lies above  $\text{HP}(\Delta)$  but for many  $\Delta$  it should be getting closer and closer to  $\text{HP}(\Delta)$  as  $p$  goes to infinity. We now make this more precise. Let  $E(\Delta)$  be the monoid generated by the lattice points in  $\Delta$ . This is a subset of  $M(\Delta)$ . Generalizing the limiting Conjecture 1.11 in [27] for  $\psi$  of order  $p$ , we have

**Conjecture 7.11.** *If the difference  $M(\Delta) - E(\Delta)$  is a finite set, then for each non-trivial  $\psi$ , we have*

$$\lim_{p \rightarrow \infty} \text{GNP}_p(\Delta, \psi) = \text{HP}(\Delta).$$

In particular,

$$\lim_{p \rightarrow \infty} \text{GNP}_p(\Delta, T) = \text{HP}(\Delta).$$

This conjecture is equivalent to the existence of the limit. This is because for all primes  $p \equiv 1 \pmod{D^*(\Delta)}$ , we already have the equality  $\text{GNP}_p(\Delta, \psi) = \text{HP}(\Delta)$  by Theorem 7.6. A stronger version of this conjecture (namely, Conjecture 1.12 in [27]) has been proved by Zhu [32] [33] [34] in the case  $m = 1$  and  $n = 1$ , see also Blache and Férard [5] [6] and Liu [21] for related further work in the case  $m = 1$  and  $n = 1$ , Hong [15] [16] and Yang [31] for more specialized one variable results. For  $n \geq 2$ , the conjecture is clearly true for any  $\Delta$  for which both  $D \leq 2$  and the Adolphson-Sperber conjecture holds, because then  $\text{GNP}_p(\Delta, \psi) = \text{HP}(\Delta)$  for every  $p > 2$ . There are many such higher dimensional examples [27]. Using free products of polytopes and the above known examples, one can construct further examples [7].

**7.3.  $T$ -adic Dwork Conjecture.** In this final subsection, we describe the  $T$ -adic version of Dwork's conjecture [10] on pure slope zeta functions.

Let  $\Lambda$  be a quasi-projective subvariety of  $N(\Delta, \mathbb{F}_p)$  defined over  $\mathbb{F}_p$ . Let  $f_\lambda$  be a family of Laurent polynomials parameterized by  $\lambda \in \Lambda$ . For each closed point  $\lambda \in \Lambda$ , the Laurent polynomial  $f_\lambda$  is defined over the finite field  $\mathbb{F}_{p^{\deg(\lambda)}}$ . The  $T$ -adic entire function  $C_{f_\lambda}(s, T)$  has the pure slope factorization

$$C_{f_\lambda}(s, T) = \prod_{\alpha \in \mathbb{Q}_{\geq 0}} P_\alpha(f_\lambda, s),$$

where each  $P_\alpha(f_\lambda, s) \in 1 + s\mathbb{Z}_p[[T]][s]$  is a polynomial in  $s$  whose reciprocal roots all have  $T^{\deg(\lambda)(p-1)}$ -slope equal to  $\alpha$ .

**Definition 7.12.** *For  $\alpha \in \mathbb{Q}_{\geq 0}$ , the  $T$ -adic pure slope  $L$ -function of the family  $f_\Lambda$  is defined to be the infinite Euler product*

$$L_\alpha(f_\Lambda, s) = \prod_{\lambda \in |\Lambda|} \frac{1}{P_\alpha(f_\lambda, s^{\deg(\lambda)})} \in 1 + s\mathbb{Z}_p[[T]][[s]],$$

where  $|\Lambda|$  denotes the set of closed points of  $\Lambda$  over  $\mathbb{F}_p$ .

The  $T$ -adic version of Dwork's conjecture is then the following

**Conjecture 7.13.** *For  $\alpha \in \mathbb{Q}_{\geq 0}$ , the  $T$ -adic pure slope  $L$ -function  $L_\alpha(f_\Lambda, s)$  is  $T$ -adic meromorphic in  $s$ .*

In the ordinary case, this conjecture can be proved using the methods in [28] [29] [30]. It would be interesting to prove this conjecture in the general case. The  $\pi_\psi$ -adic version of this conjecture is essentially Dwork's original conjecture, which can be proved as it reduces to finite rank  $F$ -crystals. The difficulty of the  $T$ -adic version is that we have to work with infinite rank objects, where much less is known in the non-ordinary case.

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